University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 10

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1. French 5-6.

(a.) All three springs are identical, with constant k. The equations of motion are

$$\frac{d^2x_A}{dt^2} + 2\omega_0^2 x_A - \omega_0^2 x_B = 0$$

$$\frac{d^2x_B}{dt^2} + 2\omega_0^2 x_B - \omega_0^2 x_A = 0$$

We plug in a guessed solution, where the two masses oscillate at the same frequency, but with different amplitudes A and B. This gives

$$-\omega^2 A + 2\omega_0^2 A - \omega_0^2 B = 0 \;\Rightarrow\; B = A\frac{2\omega_0^2 - \omega^2}{\omega_0^2}$$

$$-\omega^2 B + 2\omega_0^2 B - \omega_0^2 A = 0 \;\Rightarrow\; B = A\frac{\omega_0^2}{2\omega_0^2 - \omega^2}$$

Equating these we see that

$$(2\omega_0^2 - \omega^2)^2 - \omega_0^4 = 0$$

Solving this quadratic equation, we find that the two frequencies are $\omega^2 = \omega_0^2$ and $\omega^2 = 3\omega_0^2$.

(b.) One mass is displaced by 5 cm. This excites each normal mode equally, with amplitude 2.5 cm. To see this, first excite the first normal mode with amplitude 2.5 cm. Now both masses are +2.5 cm from equilibrium. Now excite the second normal mode, also with amplitude 2.5 cm. This moves one mass forward 2.5 cm, and the other back 2.5 cm. One is now 5cm from equilibrium, and the other is at its equilibrium position. It is mass B that is displaced, so the masses obey

$$x_A = 2.5\cos\omega_0 t - 2.5\cos\sqrt{3}\omega_0 t$$

$$x_B = 2.5\cos\omega_0 t + 2.5\cos\sqrt{3}\omega_0 t$$

(c.) After a time τ such that $\cos\omega_0\tau = \cos\sqrt{3}\omega_0\tau$, mass A returns to its equilibrium position $x_A = 0$. This happens when $\omega_0\tau$ is in the second quadrant and $\sqrt{3}\omega_0\tau$ is in the third:

$$\pi - \omega_0\tau = \sqrt{3}\omega_0\tau - \pi$$
$$\omega_0\tau = \frac{2\pi}{1+\sqrt{3}}$$

However, at $t = \tau$, mass B will not have returned to its full original |displacement| since $|\cos\tau| < 1$. Thus, even though mass A will be back in place, mass B will not, and the system will not have returned to (plus or minus) its original state. In fact, because the ratio of the two normal frequencies is irrational, once both normal modes are excited the system can never return to its original state.

2. French 5-10.

The equations of motion for this double spring system are as follows. The coordinate of the top mass is x_A and the coordinate of the bottom mass is x_B.

$$\frac{d^2x_A}{dt^2} + 2\omega_0^2 x_A - \omega_0^2 x_B = 0$$

$$\frac{d^2x_B}{dt^2} + \omega_0^2 x_B - \omega_0^2 x_A = 0$$

Plugging in the standard guess that both masses oscillate at the same frequency, but at amplitudes A and B, we get the following equations.

$$(2\omega_0^2 - \omega^2)A = \omega_0^2 B$$

$$(\omega_0^2 - \omega^2)B = \omega_0^2 A$$

Equating these, we get the quadratic equation

$$\omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$$

The solutions to this equation are

$$\omega_\pm^2 = \omega_0^2\frac{3\pm\sqrt{5}}{2}$$

The amplitudes in these modes are easily found. For ω_+, we have

$$\left(2\omega_0^2 - \frac{3}{2}\omega_0^2 - \frac{\sqrt{5}}{2}\omega_0^2\right)A_+ = \omega_0^2 B_+$$
$$B_+ = \frac{1-\sqrt{5}}{2}A_+$$

Likewise for ω_-,

$$B_- = \frac{1+\sqrt{5}}{2}A_-$$

3. French 5-14.
In the first normal mode, the three particles have an amplitude ratio $\sqrt{2}/2 : 1 : \sqrt{2}/2$. The second normal mode has amplitude ratios $1 : 0 : -1$. The third normal mode has amplitude ratios $\sqrt{2}/2 : -1 : \sqrt{2}/2$.

4. A wave is described by

$$y(x,t) = \Re\left[A_+ e^{i(\omega t - kx)} + A_- e^{i(\omega t + kx)}\right]$$

(a.) We know that the wave is moving to the left. This corresponds to the second exponential. To see this, we note that a specific place on the wave train always has the same value of $\omega t \pm kx$. We want to see what happens when t increases. For the solution $\exp(i(\omega t + kx))$, we see that as t increases, x must decrease to stay on the same place in the wave. This is a left moving wave. We thus note that $A_+ = 0$. We can now write the complex constant $A_- = Ae^{i\delta}$, where A and δ are real.

$$y(x,t) = \Re\left[Ae^{i(\omega t + kx + \delta)}\right]$$

At $x = 0$ we know that the time dependence in proportional to $\cos\omega t + \sin\omega t$. This tells us that

$$\cos(\omega t + \delta) \propto \cos\omega t + \sin\omega t$$

Using the formula for the cosine of a sum

$$\cos(\omega t + \delta) = \cos\omega t\cos\delta - \sin\omega t\sin\delta$$

For this to work we see that

$$\cos\delta = -\sin\delta \;\Rightarrow\; \delta = -\frac{\pi}{4}$$

Now we can find the amplitude and frequency. The solution is

$$y(x,t) = A\cos\left(\omega t + kx - \frac{\pi}{4}\right)$$

The amplitude is given as 0.01 m, and the period is 10^{-2} sec. The frequency $\omega = 2\pi/T = 200\pi$ sec^{-1}. The speed of waves on the string is $c =$10 m/sec, and we know that $\omega = ck$, so this tells us that $k = 20\pi$ m^{-1}. We now have the final result

$$y(x,t) = \Re\left[0.01e^{i(200\pi t + 20\pi x - \pi/4)}\right]$$
$$= 0.01\cos(200\pi t + 20\pi x - \pi/4)$$

(b.) We can now compute the maximum transverse speed and maximum slope. The transverse speed is

$$\frac{dy}{dt} = -2\pi\sin(200\pi t + 20\pi x - \pi/4)$$

The maximum value that the sine takes is 1, so the maximum transverse speed is $2\pi = 6.28$ m/sec. We can likewise find the maximum slope

$$\frac{dy}{dx} = -0.2\pi\sin(200\pi t + 20\pi x - \pi/4)$$

The maximum slope is thus $\pi/5 = 0.628$.

5. The phase velocity of surface waves is given by

$$v_{\text{ph}} = \sqrt{\frac{2\pi T}{\lambda\rho} + \frac{g\lambda}{2\pi}} = \sqrt{\frac{kT}{\rho} + \frac{g}{k}}$$

(a.) Notice that at both zero and infinite wavenumber, the phase velocity is infinite. To find the minimum phase velocity, we differentiate v_{ph} with respect to k and set to zero.

$$\frac{dv_{\text{ph}}}{dk} = \frac{T/\rho - g/k^2}{2\sqrt{kT/\rho + g/k}} = 0$$

$$k = \sqrt{\frac{\rho g}{T}} \;\Rightarrow\; \lambda = 2\pi\sqrt{\frac{T}{\rho g}}$$

The phase velocity at this wavenumber is

$$v_{\text{ph}} = \left(\frac{4gT}{\rho}\right)^{1/4}$$

The frequency $\omega = v_{\text{ph}}k$, which gives

$$\omega = \left(\frac{4gT}{\rho}\right)^{1/4}\sqrt{\frac{\rho g}{T}} = \left(\frac{4\rho g^3}{T}\right)^{1/4}$$

(b.) The group velocity of this wave is given by

$$v_{\text{gr}} = \left.\frac{d\omega}{dk}\right|_{k=\sqrt{\rho g/T}}$$

We know that $\omega = v_{\text{ph}}k$, so

$$\omega = \sqrt{\frac{Tk^3}{\rho} + gk}$$

Taking the derivative with respect to k and evaluating at the wavenumber we found before

$$\begin{aligned}\left.\frac{d\omega}{dk}\right|_{k=\sqrt{\rho g/T}} &= \left.\frac{3Tk^2/\rho + g}{2\sqrt{Tk^3/\rho + gk}}\right|_{k=\sqrt{\rho g/T}} \\ &= \sqrt{\frac{2g}{k}} = \left(\frac{4gT}{\rho}\right)^{1/4}\end{aligned}$$

The slowest waves have the same phase and group velocities. This is a general result. Look at the equation for the frequency, and differentiate it to get the group velocity

$$\omega = v_{\text{ph}}k \;\Rightarrow\; \frac{d\omega}{dk} = v_{\text{gr}} = v_{\text{ph}} + k\frac{dv_{\text{ph}}}{dk}$$

We chose the phase velocity to be a minimum, so $v_{\text{gr}} = v_{\text{ph}}$.

6. A string has tension S and linear mass density μ. This tells us the phase velocity of waves on it, $c = \sqrt{S/\mu}$. The string has length L. At $t = 0$, the string's shape is

$$y(x, 0) = 3\sin\frac{\pi x}{L} + \sin\frac{3\pi x}{L}$$

(a.) The frequencies of each of these is $\omega = ck$, so the first term has $\omega_1 = c\pi/L$ and the second term has frequency $\omega_2 = 3c\pi/L$. The periods of these two oscillations are $T_1 = 2L/c$ and $T_2 = 2L/3c$. The period of the total oscillation is the longer period, T_1. In one long period the fast oscillation has had exactly three periods. Thus, the period is

$$T = \frac{2L}{c} = 2L\sqrt{\frac{\mu}{S}}$$

(b.) After a time $T/2$, the first term has gone through a half period, and the second term has gone through one and a half periods. In both cases, this just means that there is a minus sign out front.

$$y(x, T/2) = -3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$$

7. A string of frequency 256 Hz is plucked in the exact center. This means that the even numbered modes are not exited at all. This is because the initial condition is symmetric around the middle of the string, and the even numbered modes are antisymmetric around the middle of the string. The odd numbered modes are also symmetric around the center of the string, so they survive. These frequencies are

$$f_n = (2n+1)256 \text{ Hz},\; n = 0\ldots\infty$$